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# Properties of (0, 1)-Matrices without Certain Configurations

R. P. ANSTEE*

*Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

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We generalize results of Ryser on (0, 1)-matrices without triangles, $3 \times 3$ submatrices with row and column sums 2. The extremal case of matrices without triangles was previously studied by the author. Let the row intersection of row $i$ and row $j$ ($i \neq j$) of some matrix, when regarded as a vector, have a 1 in a given column if both row $i$ and row $j$ do not 0 otherwise. For matrices satisfying some conditions on forbidden configurations and column sums $\geqslant 2$, we find that the number of linearly independent row intersections is equal to the number of distinct columns. The extremal matrices with $m$ rows and $\binom{m}{2}$ distinct columns have a unique SDR of pairs of rows with 1's. A triangle bordered with a column of 0's and its (0, 1)-complement are also considered as forbidden configurations. Similar results are obtained and the extremal matrices are closely related to the extremal matrices without triangles.

## 1. INTRODUCTION

This paper considers properties of (0, 1)-matrices which do not contain certain configurations. A *configuration* is defined to be an equivalence class of matrices where two matrices *represent* the same configuration if one matrix is a row and column permutation of the other matrix. A matrix $A$ is said to *contain* a configuration if there is a submatrix of $A$ which represents that configuration. We follow this definition of Ryser which extends the idea of submatrices [5].

Ryser considered the configuration represented by a $3 \times 3$ (0, 1)-matrix with row and column sums 2, which he called a *triangle*. Let $A^T$ denote the transpose of $A$. The following version of Helly's theorem on the real line was proven in [5]. The inequality $AA^T > 0$ requires each entry of $AA^T$ to be greater than zero.

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THEOREM 1.1 (Ryser). *Let $A$ be a* $(0,1)$*-matrix with* $AA^T > 0$ *and containing no triangles. Then $A$ has a column of* 1's.

This result can be used to prove results about the class $C(S)$ defined as follows. Let $S$ be a symmetric matrix of order $n$ with positive integral entries and 0's on the diagonal. Consider $(0,1)$-matrices $A$ with column sums at least 2. Then $A \in C(S)$ if and only if there exists a diagonal matrix $D$ such that

$$AA^T = S + D. \tag{1.1}$$

Using Theorem 1.1, Ryser proved the following result [6].

THEOREM 1.2 (Ryser). *Every matrix in the class $C(S)$ has a triangle or else the class contains exactly one matrix, apart from column permutations, without triangles.*

Using Theorem 1.2, Ryser proved the following result [7].

THEOREM 1.3 (Ryser). *Let $A$ be an $m \times n$* $(0,1)$*-matrix with column sums at least* 2, *distinct columns, and containing no triangles. Then* $n \leqslant \binom{m}{2}$.

The extremal case, $n = \binom{m}{2}$, has been studied extensively in [1]. We now consider generalizations based on replacing the triangle by a set of configurations. We prove results which are stronger than those stated in [2]. The results also appear in [3].

## 2. REPLACING THE NOTION OF A TRIANGLE

Consider the set of $(0,1)$-matrices $A$ with $k$ rows such that $AA^T > 0$ with the following minimality property. Any submatrix $B$ of $A$, obtained by deleting a column, does not have $BB^T > 0$. Let $L_k$ be the set of configurations represented by such matrices $A$. If we view the rows of the matrix as indexing vertices and the columns as complete graphs, then the elements of $L_k$ correspond to minimal edge covering of the complete graph, $K_k$, with complete subgraphs. We enumerate the representatives of $L_3$ and $L_4$.

$$L_3 = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \tag{2.1}$$

$$L_4 = \left\{ \begin{bmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&0 \\ 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \\ 0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \\ 1&1&1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For example, $L_3$ consists of the triangle and a column of 1's.

We can easily show that if $A$ is a (0, 1)-matrix of size $m \times n$ with $AA^T > 0$ and $m \geqslant k$, then any $k$ rows of $A$ contain a configuration in $L_k$. Simply let $B$ be the $k \times n$ submatrix formed by the chosen $k$ rows. We have $BB^T > 0$. We select as few columns of $B$ as possible to form a matrix $C$ with $CC^T > 0$. By definition, $C$ represents some configuration in $L_k$. Other configurations in $L_k$ may also be contained in the $k$ rows of $A$.

We now prove our generalization of Theorem 1.1. Let $J_{k,l}$ be the matrix of size $k \times l$ of all 1's.

THEOREM 2.1. *Let $A$ be a* (0, 1)*-matrix of size $m \times n$ with $AA^T > 0$ and $A$ containing no configuration in $L_k \backslash J_{k,1}$ for a given $k$ with $3 \leqslant k \leqslant m$. Then $A$ has a column of $m$* 1's.

*Proof.* We use the same slick inductive argument used in [5]. The proof is by induction on $m$ for a given $k$. The theorem is true for $m = k$ by the definition of $L_k$ and our remarks above. Assume true for $m = t \geqslant k$. Let $A$ be a (0, 1)-matrix of size $(t+1) \times n$ satisfying the hypotheses but not containing a column of 1's. Delete the first row of $A$ to obtain a submatrix $A_1$ of size $t \times n$ with $A_1 A_1^T > 0$ and $A_1$ contains no configuration in $L_k \backslash J_{k,1}$. By induction, $A_1$ has a column of $t$ 1's and so in $A$ there is a column of $t$ 1's with a zero in the first row. Repeat this for the second and third rows. Then $A$ contains the configuration represented by

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{bmatrix}. \tag{2.2}$$

Now the first $k$ rows of these three columns yield a configuration in $L_k \backslash J_{k,1}$ and hence a contradiction is reached which proves the theorem.

We note that $L_3 \backslash J_{3,1}$ consists of a triangle and so Theorem 2.1 generalizes Theorem 1.1. Also a matrix $A$, which contains no configurations in $L_k \backslash J_{k+1,1}$. The reverse implication is not always true. A few variants of Theorem 2.1 appear in [3] as well as graph theoretic interpretations of these results.

We now consider the class $C(S)$ as defined in the Introduction. We prove a generalization of Theorem 1.2 using Theorem 2.1. We use techniques slightly different from that used in [6] in order to reveal the connection with the structure of $S$. Let $S$ be of order $m$. For a square matrix $A$, we define *offdiag* $(A)$ to be the matrix obtained from $A$ by setting the diagonal entries to zero. We define the *possible blocks* of $S$ to be the matrices $B$ of order $m$

with $B = \text{offdiag}(\beta\beta^T)$ and $B \leqslant S$, where $\beta$ is a (0, 1)-column vector of length $m$. We recall that for $A = (a_{ij})$ and $C = (c_{ij})$, we say $A \leqslant C$ if and only if $a_{ij} \leqslant c_{ij}$ for all pairs $i, j$. We will be considering maximal possible blocks in this ordering. The *size* of the block $B$ is defined to be the number of 1's in $\beta$. Let $A = \text{offdiag}(\alpha\alpha^T)$ be a possible block of $S$. Then $\alpha$ would be a column of some matrix in $C(S)$. Simply take a matrix $A' \in C(S - A)$ and append $\alpha$ to $A'$ to get a matrix $A'' \in C(S)$.

THEOREM 2.2. *For every matrix $A \in C(S)$ containing no configurations in $L_k \backslash J_{k,1}$, the columns with column sum at least $k$ are unique apart from order.*

*Proof.* Take an arbitrary matrix $A \in C(S)$ with no configuration in $L_k \backslash J_{k,1}$. Let $B_1 = \text{offdiag}(\beta\beta^T)$ be a maximal possible block of $S$ of size at least $k$. Let $\beta$ have 1's in rows $i_1, i_2, \ldots, i_r$. The rows $i_1, i_2, \ldots, i_r$ of $A$ form a submatrix $C$ with $CC^T > 0$. Certainly $C$ has no configurations in $L_k \backslash J_{k,1}$, since $A$ does not, and thus by Theorem 2.1, $C$ has a column of 1's. Thus $\beta$ is a column of $A$ using the maximality of $B_1$. Delete $\beta$ from $A$ to form $A_1$ and set $S_1 = S - B_1$. Then $A_1 \in C(S_1)$. Now $A_1$ has no configurations in $L_k \backslash J_{k,1}$ and thus we repeat the process on $A_1$. We continue this until, at the $t$th stage, there is no maximal possible block of $S_t = S - B_1 - B_2 - \cdots - B_t$ of size at least $k$. Thus any matrix $A_t \in C(S_t)$ has column sums all less than $k$.

We see that the columns of $A$ with column sum at least $k$ are precisely $\beta_1, \beta_2, \ldots, \beta_t$, where $B_i = \text{offdiag}(\beta_i\beta_i^T)$. The selection of $B_i$'s depends only on $S$. Thus for any matrix in $C(S)$, with no configurations in $L_k \backslash J_{k,1}$, the columns of column sum at least $k$ are precisely $\beta_1, \beta_2, \ldots, \beta_t$, proving the theorem.

The case $k = 3$ is the case of matrices containing no triangles. We note that the columns of column sum 2 are specified uniquely once the columns of column sum at least 3 are specified. Thus Theorem 2.2 generalizes Theorem 1.2. In the proof, we have shown that the selection of the $B_i$'s is unique. Let $B_1, B_2, \ldots, B_t$ be a *sequence of maximal possible blocks* of $S$, where $B_i = \text{offdiag}(\beta_i\beta_i^T)$ is a maximal possible block of $S - B_1 - B_2 - \cdots - B_{i-1}$ of size at least $k$ for $i = 1, 2, \ldots, t$. In addition, we assume that $S - B_1 - B_2 - \cdots - B_t$ has no possible blocks of size at least $k$.

THEOREM 2.3. *There exists a matrix $A \in C(S)$, where $A$ contains no configurations in $L_k \backslash J_{k,1}$ if and only if the sequence of maximal blocks $B_1, B_2, \ldots, B_t$ is unique apart from order.*

*Proof.* Let $A \in C(S)$, where $A$ contains no configuration in $L_k \backslash J_{k,1}$. A sequence of maximal blocks $B_1, B_2, \ldots, B_t$ as defined above tells us using the proof of Theorem 2.2, that $\beta_1, \beta_2, \ldots, \beta_t$ are the columns of $A$ of column sum at least $k$. These columns are unique apart from order and so $B_1, B_2, ..., B_t$ are unique apart from order.

Assume that the sequence of maximal blocks is unique apart from order. Let $S_t = S - B_1 - B_2 - \cdots - B_t$ and let $A_t \in C(S_t)$. Form a matrix $A$ by appending to $A_t$ the columns $\beta_1, \beta_2, ..., \beta_t$. Assume $A$ has a submatrix $B$, in rows $i_1, i_2, ..., i_k$, which represent a configuration in $L_k \backslash J_{k,1}$. Let those columns of $A$ which have 1's in all the rows $i_1, i_2, ..., i_k$ be $\alpha_1, \alpha_2, ..., \alpha_r$ and let $A_i = \text{offdiag}(\alpha_i \alpha_i^T)$. Then consider the matrix $C$ obtained from $A$ be deleting the columns $\alpha_1, \alpha_2, ..., \alpha_r$. Now $B$ does not have a column of 1's, thus $B$ is a submatrix of $C$. Let $S' = S - A_1 - A_2 - \cdots - A_r$ and then $C \in C(S')$. Since $BB^T > 0$ we deduce that $S'$ has a possible block $B' = \text{offdiag}(\alpha\alpha^T)$, where $\alpha$ has 1's in rows $i_1, i_2, ..., i_k$. Thus there is a maximal possible block $A_{r+1} = \text{offdiag}(\alpha_{r+1}\alpha_{r+1}^T)$ of $S'$, where $\alpha_{r+1}$ has 1's in rows $i_1, i_2, ..., i_k$. Completing $A_1, A_2, ..., A_r, A_{r+1}$ to a sequence of maximal possible blocks of $S$ contradicts the uniqueness of such sequences. Thus $A$ contains no configurations in $L_k \backslash J_{k,1}$ as desired.

## 3. The Row Intersection Theorem

We are able to generalize Theorem 1.3 using similar proof techniques. The condition "contains no triangles" is replaced by the following. Define CONDITION to be the condition "for $i \geqslant 3$, contains no configurations in

$$\{[J_{i,1} C] \mid C \in L_i \backslash J_{i,1}\}, \tag{3.1}$$

in those columns with column sum at most $i$." For example, the set in (3.1) for $i = 3$ is the single configuration represented by

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{3.2}$$

For $i = 4$, we obtain the set

$$\left\{ \begin{bmatrix} 1&1&1&1&0&0&0 \\ 1&1&0&0&1&1&0 \\ 1&0&1&0&1&0&1 \\ 1&0&0&1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&1&0 \\ 1&1&0&0&1 \\ 1&0&1&0&1 \\ 1&0&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&0 \\ 1&1&0&1 \\ 1&0&1&1 \\ 1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&0 \\ 1&1&0&1 \\ 1&0&1&1 \\ 1&1&1&1 \end{bmatrix} \right\}. \tag{3.3}$$

The condition "contains no configuration (3.2)" is stronger (more restrictive) than CONDITION. The condition "contains no triangles" is stronger still and so either can be substituted for CONDITION in the following results. We define the *row intersection* of row $i$ and row $j$ $(i \neq j)$, regarded as a vector, to have a 1 in a column if both row $i$ and row $j$ do and a 0 otherwise. This is the Hadamard product of the two rows.

THEOREM 3.1. *Let $A$ be a $(0, 1)$-matrix of size $m \times n$ with column sums at least 2 and satisfying* CONDITION. *Then the number of linearly independent row intersections (over the rationals) is equal to the number of distinct columns of $A$.*

*Proof.* We follows the proof techniques used in [7, 8]. Let $A$ be a $(0, 1)$-matrix of size $m \times n$. We can view $A = (a_{ij})$ as indexing $m$ subsets of an $n$-set. Let the sets by $S_1, S_2, \dots, S_m$ and the $n$-set be $\{x_1, x_2, \dots, x_n\}$. We let $x_j \in S_i$ if and only if $a_{ij} = 1$. Ryser introduced the fundamental matrix equation for finite sets. Let $X = \operatorname{diag}(x_1, x_2, \dots, x_n)$ which is the diagonal matrix with entries $x_1, x_2, \dots, x_n$ on the diagonal. For this purpose we consider $x_1, x_2, \dots, x_n$ as independent indeterminates. Let

$$AXA^T = Y. \tag{3.4}$$

The matrix $Y$ contains a great deal of information about the sets. Let $Y = (y_{ij})$. Then $y_{ij}$ is the sum of the elements of $S_i \cap S_j$. We note that the row intersections, as defined above, are precisely these $y_{ij}$'s when thought of as vectors in $Q^n$ with basis $\{x_1, x_2, \dots, x_n\}$.

Repeated columns can be deleted without affecting the linear independence of the row intersections so we will assume that $A$ has no repeated columns; i.e., $n$ is the number of distinct columns. We have immediately that $n$ is greater than or equal to the number of linearly independent row intersections since $n$ is the dimension of the space which contains the row intersections.

Assume $n$ is greater than the number of linearly independent row intersections. We now consider $x_1, x_2, \dots, x_n$ as variables and solve for $y_{ij} = 0$ $(i \neq j)$. Since the number of variables exceeds the number of linearly independent equations, we can find rational and hence integral values $e_1, e_2, \dots, e_n$, not all zero, for $x_1, x_2, \dots, x_n$. Let $E = \operatorname{diag}(e_1, e_2, \dots, e_n)$. Then

$$AEA^T = D, \tag{3.5}$$

where $D$ is a diagonal matrix. Every variable occurs in some equation (column sums at least 2), thus some $e_i$'s are positive and some are negative. Define $A_1$ and $A_2$ as follows. For all $i$ with $e_i > 0$, $A_1$ is to contain column $i$ of $A$ repeated $e_i$ times. For all $j$ with $e_j < 0$, $A_2$ is to contain column $j$ of $A$ repeated $-e_j$ times. Then

$$A_1A_1^T - A_2A_2^T = D, \tag{3.6}$$

and so $A_1, A_2 \in C(S)$ for some $S$. Now we apply Theorem 2.1. Among all the columns of $A_1$ and $A_2$, let $\alpha$ be the column of largest column sum and have it occur in $A_1$. Let $\alpha$ have $i$ 1's. If $i=2$, then $A_1$ and $A_2$ are easily seen to be equal apart from a column permutation. They are the canonical matrices for $C(S)$ as defined in [6]. For $i>2$, let the 1's of $\alpha$ be in rows $j_1, j_2, \ldots, j_i$. Let $B$ be the submatrix of $A_2$ consisting of rows $j_1, j_2, \ldots, j_i$ of $A_2$. Now $A_1, A_2 \in C(S)$ and $A_1 A_1^T \geqslant \alpha\alpha^T$. Thus $BB^T > 0$. If $B$ contains a configuration $C$ in $L_i \backslash J_{i,1}$, then appending it to the $i$ 1's of $\alpha$ yields a configuration in (3.1) in columns of column sum at most $i$. This violates CONDITION. Repeated columns do not cause problems since no configuration in $L_i$ has repeated columns. Thus $B$ has no configuration in $L_i \backslash J_{i,1}$ and hence, by Theorem 2.1, $B$ has a column of 1's. Thus $\alpha$ is also a column of $A_2$. We could continue to show that $A_1$ and $A_2$ are the same apart from a column permutation but we need only note that this forces a repeated column in $A$. This is a contradiction which proves that $n$, the number of distinct columns, is precisely the number of linearly independent row intersections.

COROLLARY 3.2. *Let $A$ be a $(0, 1)$-matrix of size $m \times n$ with column sums at least 2, distinct columns, satisfying* CONDITION. *Then $n \leqslant \binom{m}{2}$.*

*Proof.* This follows directly from Theorem 3.1 by noting that there are only $\binom{m}{2}$ row intersections.

This yields Theorem 1.3 by replacing CONDITION by the condition "contains no triangles." Using quite direct means, Cunningham [4] obtained this bound in the case that CONDITION is replaced by the condition "contains no configurations (3.2)."

## 4. UNIQUE SDR PROPERTY FOR SPECIAL MATRICES

Corollary 3.2 leads one to consider the extremal case when $n=\binom{m}{2}$. We will call a (0, 1)-matrix $A$ *special* if $A$ is of size $m \times \binom{m}{2}$ with column sums at least 2, distinct columns and satisfying CONDITION. We obtain a nice structure result for special matrices. Define $K_m$ as a (0, 1)-matrix of size $m \times \binom{m}{2}$ with all possible columns of column sum 2. One ordering of the columns is selected. Associate when each column $i$, of a special matrix $A$, the set of pairs

$$S_i = \{\{j, k\} \mid \text{column } i \text{ has 1's in rows } j \text{ and } k \ (j \neq k)\}. \tag{4.1}$$

Then searching for a permutation matrix $P$ of order $\binom{m}{2}$ with $A \geqslant K_m P$ is equivalent to looking for a system of distinct representatives (SDR). The following lemma will be used to show the existence of a unique matrix $P$.

LEMMA 4.1. *Let $A$ be a $(0, 1)$-matrix of size $m \times n$ with column sums at* *least* 2, *distinct columns, and satisfying* CONDITION. *Then there is an off diagonal entry of* $AA^T$ *equal to* 1.

*Proof.* Assume there is no off diagonal entry of $AA^T$ equal to 1. Let $\alpha$ be the column in $A$ with the largest column sum, say, $k$, with 1's in rows $i_1, i_2, \ldots, i_k$. If $k=2$, distinct columns force all the nonzero off diagonal entries of $AA^T$ to be 1. We assume $k \geqslant 3$. Let $A_1$ be the matrix consisting of rows $i_1, i_2, \ldots, i_k$ of $A$. Then $A_1 A_1^T \geqslant 2J_k$, where $J_k$ is the matrix of order $k$ of all 1's. Let $A_2$ be the matrix obtained from $A_1$ by deleting the column of all 1's which comes from $\alpha$. Then $A_2 A_2^T \geqslant J_k > 0$ and so $A_2$ has a configuration in $L_k$. This configuration cannot be a column of 1's since $A$ has distinct columns. Thus CONDITION is violated with a configuration in (3.1) for $i=k$ in columns of column sum at most $k$. Thus $AA^T$ has an off diagonal entry equal to 1.

THEOREM 4.2. *Let $A$ be a special matrix. Then there exists a unique permutation matrix $P$ of order $\binom{m}{2}$ such that $A \geqslant K_m P$.*

*Proof.* We define the sets $S_i$ as in (4.1). If $AA^T$ has a 1 in position $(j, l)$ $(j \neq l)$, where column $i$ has 1's in rows $j$ and $l$ then we select $\{j, l\}$ from $S_i$. We note that $S_i$ is the only set containing $\{j, l\}$ and so this gives uniqueness. We delete column $i$ from $A$ to obtain a matrix $A_1$. We repeat the process on $A_1$until the SDR is complete. Lemma 4.1 ensures that we can continue at each stage. Uniqueness follows from the construction.

There is an interesting interplay between Lemma 4.1 and Theorem 3.1. Using the selection process outlined above, write out the row intersections as they are chosen. For example, in the above proof we chose the row intersection of row $j$ and row $l$ first. One immediately deduces that the row intersections so chosen are linearly independent and so Theorem 3.1 follows. In contrast, no way has been found to avoid the fundamental matrix equation in Theorem 5.2.

PROPOSITION 4.3. *Let $A$ be a* (0, 1)*-matrix of size $m \times \binom{m}{2}$ such that there is a unique permutation matrix $P$ of order $\binom{m}{2}$ with $A \geqslant K_m P$. Then $A$ has at most $\binom{m}{2} + \binom{m}{3}$ 1's.*

*Proof.* We assume $P = I$ by replacing $A$ by $AP^{-1}$. $I$ denotes the identity matrix. Thus $A$ can be viewed as formed from $K_m$ by adding 1's. We recall that in $K_m$, every possible column of column sum 2 occurs precisely once. We claim that any three rows of $K_m$ contain a unique triangle. Let $i_1, i_2, i_3$ be three rows of $K_m$. Then the unique triangle in these three rows is found in the three columns which have two 1's in these three rows.

We claim that each 0 of $K_m$ is contained in one of the $\binom{m}{3}$ triangles. Say the 0 occurs in row $i_1$ and in a column with 1's in row $J$ in $i_2$ and $i_3$. These two 1's must be in the triangle and thus our claim follows using the first claim. At most one 1 can be added to a triangle in $K_m$ and still keep the matrix $P$ unique. Thus the result follows.

This bound is achieved for matrices without triangles since each of the $\binom{m}{3}$ triangles of $K_m$ must be "killed" by adding a 1. A converse to Theorem 4.2 using this bound does not hold. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}. \tag{4.2}$$

This matrix satisfies the hypotheses and bound of Proposition 4.3, yet the last four columns and last three rows violate CONDITION.

## 5. An Application of Another Configuration Theorem of Ryser

In the proof of Theorem 3.1, we set $y_{ij} = 0$ for $i \neq j$ in the fundamental matrix equation. If we set $y_{ij} = 0$ for all pairs $i, j$ we might end up with matrices $A_1, A_2$ from $A$ with

$$A_1 A_1^T - A_2 A_2^T = 0, \tag{5.1}$$

where 0 is the matrix of 0's. We need a new configuration theorem to handle this. Ryser proved the following result [5].

Theorem 5.1 (Ryser). *Let $A$ and $B$ be $(0, 1)$-matrices of size $m \times n$ such that $AA^T = BB^T$ and $A$ contains no configurations represented by*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{5.2}$$

*Then $A$ and $B$ are the same apart from a column permutation.*

We prove the following variation, which is more useful for our purposes.

Theorem 5.2. *Let $A$ and $B$ be $(0, 1)$-matrices with column sums at least 1 such that $AA^T = BB^T$ and $A$ contains no configurations represented by*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{5.3}$$

*Then $A$ and $B$ are the same apart from a column permutation.*

*Proof.* Assume $A$ is of size $m \times n$ and $B$ is of size $m \times (n+l)$ with $l \geqslant 0$. Form the matrix $A_1$ from $A$ by adding $l$ columns of all 0's. Then $A_1A_1^T = AA^T = BB^T$. Certainly $A_1$ has no configuration (5.3) since $A$ does not. We use Ryser's proof of Theorem 5.1 with $A_1$ replacing $A$. His proof dealt with the transposes of the matrices. After deleting columns from $A_1$ and $B$ that match we obtain matrices $A^*, B^*$. There are six possible rows formed by four columns selected from $A^*$ and $B^*$. We omit further details and refer the reader to [5].

$$A^*: \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}, \qquad B^*: \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}. \tag{5.4}$$

The first and last rows occur precisely once by construction. If the third row occurs, the $A^*$ contains the forbidden configuration (5.3). If the fourth row occurs, then $B^*$ contains the forbidden configuration (5.3). Thus neither the third or fourth rows occur. Then the first column of $A^*$ and the fourth column of $B^*$ match in (5.4). This is a contradiction and so $A^*, B^*$ are null.

In our case, we now have $A_1$ and $B$ equal apart from a column permutation. Thus $B$ has $l$ columns of 0's and so $l = 0$ and $A = A_1$.

We state the companion result without proof. It is not quite the same because appending a column of 0's may create the forbidden configuration (5.5).

THEOREM 5.3. *Let $A$ and $B$ be* (0, 1)*-matrices of size $m \times n$ such that $AA^T = BB^T$ and $A$ and $B$ have no configuration represented by*

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{5.5}$$

*Then $A$ and $B$ are the same apart from a column permutation.*

We now apply Theorem 5.2.

THEOREM 5.4. *Let $A$ be a* (0, 1)*-matrix of size $m \times n$ with column sums at least* 1 *and $A$ has no configuration* (5.3). *Then the number of linearly independent row intersections and rows* (*over the rationals*) *is equal to the number of distinct columns of $A$.*

*Proof.* We follow the proof of Theorem 3.1 with a few alterations. We note that the entries $y_{il}$, when considered as vectors in $Q^n$ with basis

$\{x_1, x_2,\dots, x_n\}$, correspond to the rows. We may assume that $A$ has no repeated columns. Thus $n$, the number of distinct columns, is greater than or equal to the number of linearly independent row intersections and rows since $n$ is the dimension of the space containing them. We proceed under the assumption that $n$ is greater than the number of linearly independent row intersections and rows. Set $y_{ij}=0$ for all pairs $i, j$. The result is a pair of matrices $A_1, A_2$ obtained from disjoint sets of columns of $A$ with

$$A_1A_1^T - A_2A_2^T = 0. \tag{5.6}$$

Since $A_1$ and $A_2$ contain no configuration (5.3), we apply Theorem 5.2 and obtain that they are equal apart from a column permutation. Thus $A$ has a repeated column, which is a contradiction. This proves the theorem.

COROLLARY 5.5. *Let $A$ be a* (0, 1)*-matrix of size $m \times n$ with column sums at least* 1, *distinct columns, and containing no configuration* (5.3). *Then $n \leqslant \binom{m+1}{2}$.*

*Proof.* This follows directly from Theorem 5.4 by noting that the total number of row intersections and rows is $\binom{m}{2} + m = \binom{m+1}{2}$.

## 6. Structure Results for Restricted Matrices

Corollary 5.5 leads one to consider the extremal case when $n = \binom{m+1}{2}$. We call a (0, 1)-matrix $A$ *restricted* if $A$ is of size $m \times \binom{m}{2}$ with column sums at least 1, distinct columns, and containing no configuration (5.3). Consider any $r$ rows of $A$. All the $\binom{r+1}{2}$ row intersections and rows are linearly independent. Let $B$ be the submatrix of $A$ consisting of these $r$ rows. Deleting columns of 0's or repeated columns will not affect the number of linearly independent row intersections and rows. Thus by Theorem 5.4, we deduce that the submatrix $C$ that results is of size $r \times \binom{r+1}{2}$ and so is a restricted matrix. We have shown that any $r$ rows of $A$ have a submatrix which is a restricted matrix of size $r \times \binom{r+1}{2}$.

In analogy to Theorem 4.2, we may prove the following.

THEOREM 6.1. *Let $A$ be a restricted matrix. Then there exists a permutation matrix $P$ of order $\binom{m+1}{2}$ such that $A \geqslant [K_m I_m]\, P$.*

*Proof.* Associate with each column of $A$ the set

$$S_i = \{\{j, k\} \mid \text{column } i \text{ has 1's in rows } j \text{ and } k\}, \tag{6.1}$$

where we allow $j = k$. Finding $P$ is equivalent to finding as SDR for this system of sets. Let $I \subseteq \{1, 2,\dots, \binom{m+1}{2}\}$ and $|I| = r$. Let $B$ be the $m \times r$ matrix obtained by selecting from $A$ precisely the columns given by $I$. Then in $\bigcup_{i\in I} S_i$, the number of elements is equal to the number of nonzero row intersections and rows in $B$ which is at least the number of linearly independent row intersections and rows which is equal to $r$ by Theorem 5.4. But then $|\bigcup_{i\in I} S_i| \geqslant |I|$ for all $I$. Hence by P. Hall's SDR theorem, the SDR exists and so $P$ exists.

We may classify all restricted matrices. We recall that in [1], a *solution* was defined to be a (0, 1)-matrix of size $m \times \binom{m}{2}$, with column sums at least 2, distinct columns, and no triangles. We define the (0, 1)-*complement* of a matrix to be the matrix obtained by replacing 1's by 0's and vice versa.

THEOREM 6.2. *Let $A$ be a restricted matrix, then after a suitable row and column permutation, $A = [K_m I_m]$ or*

$$A = [J_{1,m} | J_m - I_m | A'], \tag{6.2}$$

*where $A'$ is the* (0, 1)*-complement of a solution with the resulting column of* 0*'s deleted.*

*Proof.* We consider the effects of various column sums. If $A$ has column sums at most 2, then it is a column permutation of $[K_m I_m]$. If $A$ is restricted and has a column of 1's, then consider $A_1$, its (0, 1)-complement. It has no configuration (5.5) and so has no triangles since it has a column 0's. There are at most $m+1$ columns of column sum 0 or 1 in $A_1$ and $A_1$ has no column of 1's. Thus there are $\binom{m}{2} - 1$ columns of column sum at least 2 and so they form a solution apart from a missing column of 1's. Hence $A$ is in the form given in (6.2).

We prove the result by induction on $m$. For $m = 2, 3$, the theorem follows from the above remarks. For $m = 4$, we are done if either all the columns have column sum at most 2 or there is a column of column sum 4. The remaining case yields only one possible matrix,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6.3}$$

But this has configuration (5.3) in rows 1, 2, 4 and columns 3, 4, 6, 10. This verifies the theorem for $m = 4$.

Assume the theorem is true for $m = k$ ($k \geqslant 4$). Let $A$ be a restricted matrix of size $(k+1) \times \binom{k+2}{2}$. Any $k$ rows of $A$ have a restricted $B$ as a submatrix of size $k \times \binom{k+1}{2}$. By induction, $B$ is of one of the two forms given.

If $B = [K_k I_k]$, then $A$ has column sums at most 3. Thus any $k$ rows of $A$ contain $[K_k I_k]$ using the fact that $k \geqslant 4$ and noting that (6.2) has a column of $k$ 1's. Thus $A$ has column sums at most 2 and so after a column permutation, $A = [K_{k+1} I_{k+1}]$.

If $B$ has the form given in (6.2), then $A$ has a column of $k$ 1's and so any submatrix of $A$, consisting of $k$ rows of $A$, has a column of at least $k-1$ 1's. Since $k-1 \geqslant 3$, for any $k$ rows, $B$ has the form given in (6.2). Thus for any three rows, there is a column in $A$ with 1's in these rows and this implies $A$ has no configuration represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.4}$$

By maximality, $A$ has a column of 1's and thus $A$ is in the form given in (6.2).

Noting that (5.3) is the (0, 1)-complement of (5.5), then the (0, 1)-complement of any matrix, without a configuration (5.5), has no configuration (5.3). Utilizing Theorem 6.2, we obtain the following result where we allow a column of 0's.

THEOREM 6.3. *Let $A$ be a* (0, 1)*-matrix of size $m \times n$ with distinct columns and no configurations* (5.5). *Then $n \leqslant \binom{m+1}{2} + 1$. In the case of equality, after a column permutation, $A = [J_{m,1} K']$, where $K'$ is the* (0, 1)*-complement of $[K_m I_m]$, or $A = [B I_m O_{m,1}]$, where $B$ is a solution of size $m \times \binom{m}{2}$ and $O_{m,1}$ is the column of 0's.*

The latter case is particularly interesting. Define column intersections in the same way as row intersections. From our results in [1], then any column intersection is also a column of the matrix. The columns form an intersection closed set.

As a final note, having classified the extremal matrices, we see that a variant of Lemma 4.1 does not hold here. Thus the use of the fundamental matrix equation seems vital.

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